

Fore-and-Aft Translations from FOL into Coherent Logic

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Abstract

This note describes an intriguing method for translating FOL into the *geolog* language for coherent logic. The method systematically attempts to translate occurrences of formal implication in FOL inputs into coherent logic rules containing implication in a “forward” direction, rather than expressing formal implication using a disjunction, which involves a “splitting” or “case” translation. The advantage of forward translation is a reduced search space when the resulting rules are used to compute consequences: The forward rules maintain the current branch deduction, whereas a splitting translation promotes new branches.

1 Background information and motivating examples

An elegant direct method for a translation from FOL (*first-order logic*) into coherent logic was outlined in [1]. The method consists of encoding the semantic tableaux proof method into a set of coherent rules, that is, a *geolog theory* that can be computed by a *Skolem Machine* [2]. See reference [3] for more information about FOL translation, another characterization of the basic translation method, and an important detailed explanation for why these translations yield *correct proof objects*, as opposed to other FOL translation approaches which do not.

Andrew Polonsky has worked on promising *heuristic* translation methods that “favor” forward translation, using specific contrapositive forms of disjunctive coherent rules (discussed briefly in [3]).

This note specifies a *deterministic* algorithm for forward translation that arises from formal implication. Polonsky’s approach is likely to be more general because it uniformly attempts to use contrapositive forms for all disjunctions— not just those which arise from formal implications.

In order to motivate this approach, we present several telling examples.

Example 1. Suppose that $\phi = ((a \vee b) \rightarrow c) \rightarrow d$ is an axiom. The ϕ is converted to geolog rules, as follows:

$$\begin{aligned}
 true &\Rightarrow \overrightarrow{((a \vee b) \rightarrow c) \rightarrow d}. \\
 \overrightarrow{((a \vee b) \rightarrow c) \rightarrow d}, \overleftarrow{((a \vee b) \rightarrow c)} &\Rightarrow \overrightarrow{d}. \\
 \overleftarrow{\neg(a \vee b)} &\Rightarrow \overleftarrow{((a \vee b) \rightarrow c)}. \\
 \overleftarrow{c} &\Rightarrow \overleftarrow{((a \vee b) \rightarrow c)}. \\
 \overleftarrow{\neg a}, \overleftarrow{\neg b} &\Rightarrow \overleftarrow{\neg(a \vee b)}. \\
 not_a &\Rightarrow \overleftarrow{\neg a}. \\
 not_b &\Rightarrow \overleftarrow{\neg b}. \\
 c &\Rightarrow \overleftarrow{c}. \\
 \overrightarrow{d} &\Rightarrow d.
 \end{aligned}$$

In addition, we always have consistency axioms. Explicit negation of a predicate p is expressed here as not_p .

$$a, not_a \Rightarrow false.$$

$$b, not_b \Rightarrow false.$$

$$c, not_c \Rightarrow false.$$

$$d, not_d \Rightarrow false.$$

The *fore operator* $\overrightarrow{\cdot}$ is *tried* whenever formal implication is encountered. The trial fails when an exceptional case is encountered while trying to proceed in the fore direction. In this example *fore translation* goes smoothly, without exception. The *aft operator* $\overleftarrow{\cdot}$ always applies. Example 3 below will illustrate how fore translation can lead to an exceptional case, and the examples explain how to translate when an exception arise.

The next example illustrates some fore and aft translation patterns using quantified variables. Again, the fore translation goes smoothly.

Example 2. Suppose that $\phi = (\forall x)(\neg p(x) \rightarrow (\exists y)(\exists z)q(x, u, z))$.

$$\begin{aligned} true &\Rightarrow \overrightarrow{(\forall x)(\neg p(x) \rightarrow (\exists y)(\exists z)q(x, u, z))}. \\ \overrightarrow{(\forall x)(\neg p(x) \rightarrow (\exists y)(\exists z)q(x, u, z))}, dom(X) &\Rightarrow \overrightarrow{\neg p(x) \rightarrow (\exists y)(\exists z)q(x, u, z)}(X). \\ \overrightarrow{\neg p(x) \rightarrow (\exists y)(\exists z)q(x, u, z)}(X), \overleftarrow{\neg p(x)}(X) &\Rightarrow \overrightarrow{(\exists y)(\exists z)q(x, u, z)}(X). \\ not_p(X) &\Rightarrow \overleftarrow{\neg p(x)}(X). \\ \overrightarrow{(\exists y)(\exists z)q(x, u, z)}(X) &\Rightarrow dom(Y), \overrightarrow{(\exists z)q(x, u, z)}(X, Y). \\ \overrightarrow{(\exists z)q(x, u, z)}(X, Y) &\Rightarrow dom(Z), \overrightarrow{q(x, u, z)}(X, Y, Z). \\ \overrightarrow{q(x, u, z)}(X, Y, Z) &\Rightarrow q(X, Y, Z). \end{aligned}$$

The geolog rules are verbosely expressed in order to illustrate translation steps incrementally. Automated translators can skip or combine several steps without loss of generality. Also, the variable correspondences are presented in an intuitive fashion (e.g., formal variable x becomes X in the geolog rule).

The next example illustrates an exceptional case of fore translation. The corresponding (embedded) formal implication will have to be expressed using disjunction in a coherent rule.

Example 3. Suppose that $\phi = (\forall x)((\forall y)p(x, y) \rightarrow q(x))$. Note that there is an embedded universal quantifier in the antecedent of the embedded formal implication. First, let us see how the exception to fore translation arises ...

$$\begin{aligned} true &\Rightarrow \overrightarrow{(\forall x)((\forall y)p(x, y) \rightarrow q(x))}. \\ \overrightarrow{(\forall x)((\forall y)p(x, y) \rightarrow q(x))}, dom(X) &\Rightarrow \overrightarrow{((\forall y)p(x, y) \rightarrow q(x))}(X). \\ * \overrightarrow{((\forall y)p(x, y) \rightarrow q(x))}(X), \overleftarrow{(\forall y)p(x, y)}(X) &\Rightarrow \overrightarrow{q(x)}(X). \end{aligned}$$

EXCEPTIONAL : ...?? $\Rightarrow \overleftarrow{(\forall y)p(x,y)}(X)$.

so * is replace by ...

$\overrightarrow{((\forall y)p(x,y) \rightarrow q(x))}(X) \Rightarrow \overrightarrow{\neg(\forall y)p(x,y)}(X), \overrightarrow{q(x)}(X)$.

and continue ...

$\overrightarrow{\neg(\forall y)p(x,y)}(X) \Rightarrow \overrightarrow{(\exists y)\neg p(x,y)}(X)$.

$\overrightarrow{(\exists y)\neg p(x,y)}(X) \Rightarrow \text{dom}(Y), \overrightarrow{\neg p(x,y)}(X, Y)$.

$\overrightarrow{\neg p(x,y)}(X, Y) \Rightarrow \text{not_}p(X, Y)$.

In this example, the attempt to use a fore translation for the embedded formal implication leads to an exception: There is no fore translation for a universal form. Thus the translation must be redone with aft translation.

2 Formal schemas defining fore-and-aft translations

Let us assume that the FOL theory has zero or more *axioms* of the form $\top \rightarrow \alpha$ (affirmation) and one *conjecture* of the form $\gamma \rightarrow \perp$ (denial).

The axioms and conjecture are translated into geolog rules. An axiom $\top \rightarrow \alpha$ is translated into the geolog rule $\text{true} \Rightarrow \overrightarrow{\alpha}$ (affirm axiom) together with translations for $\overrightarrow{\alpha}$. The conjecture $\gamma \rightarrow \perp$ is translated into geolog rule $\text{true} \Rightarrow \overrightarrow{\neg\gamma}$ (denial of conjecture) together with translations for $\overrightarrow{\neg\gamma}$.

The right-facing arrow \Rightarrow indicates the ‘‘aft’’ direction for translation. A so-called ‘‘fore’’ direction \Leftarrow will arise upon consideration of the detailed translations for the $\overrightarrow{\alpha}$ (axiom) or $\overrightarrow{\neg\gamma}$ (conjecture) forms, and this is what we now specify.

Fig. 1 specifies the *aft* translations. These are essential the same as those specified in Reference [1] except for schema (4). Notice that schema (4) is the only aft schema to introduce a *fore* translation. If the fore translation encounters an exception then one uses the aft schema to translate the form. Fig. 2 specifies the fore translations.

In the schemas, \hat{x} always denotes an ordered sequence of variable arguments that arose *before* the current schema is supposed to apply. For convenience, we assume that the lower-case variable letters in the FOL wffs become upper-case when the corresponding variable gets elevated to arguments in the geolog rules. (There is nothing special about this convention other than easier reading.)

- (1) $\overrightarrow{\phi \wedge \psi}(\hat{x}) \Rightarrow \overrightarrow{\phi}(\hat{x}), \overrightarrow{\psi}(\hat{x})$
- (2) $\overrightarrow{\phi \vee \psi}(\hat{x}) \Rightarrow \overrightarrow{\phi}(\hat{x}) | \overrightarrow{\psi}(\hat{x})$
- (3) a) $\overrightarrow{\neg(\phi \wedge \psi)}(\hat{x}) \Rightarrow \overrightarrow{\neg\phi}(\hat{x}) | \overrightarrow{\neg\psi}(\hat{x})$
 b) $\overrightarrow{\neg(\phi \vee \psi)}(\hat{x}) \Rightarrow \overrightarrow{\neg\phi}(\hat{x}), \overrightarrow{\neg\psi}(\hat{x})$
 c) $\overrightarrow{\neg\neg\phi} \Rightarrow \overrightarrow{\phi}$
 d) $\overrightarrow{\neg(\phi \rightarrow \psi)}(\hat{x}) \Rightarrow \overrightarrow{\phi}(\hat{x}), \overrightarrow{\neg\psi}(\hat{x})$
 e) $\overrightarrow{\neg(\forall \hat{x})\phi}(\hat{y}) \Rightarrow \overrightarrow{(\exists \hat{x})\neg\phi}(\hat{y})$
 f) $\overrightarrow{\neg(\exists \hat{x})\phi}(\hat{y}) \Rightarrow \overrightarrow{(\forall \hat{x})\neg\phi}(\hat{y})$
 g) $\overrightarrow{\neg a(\hat{x})} \Rightarrow \text{not_}a(\hat{X})$ when a is atomic
- (4) $\overrightarrow{\phi \rightarrow \psi}(\hat{x}), \overleftarrow{\phi}(\hat{x}) \Rightarrow \overrightarrow{\psi}(\hat{x})$ if translation of $\overleftarrow{\phi}$ is *not exceptional*
 $\overrightarrow{\phi \rightarrow \psi}(\hat{x}) \Rightarrow \overrightarrow{\neg\phi} | \overrightarrow{\psi}(\hat{x})$ otherwise
- (5) $\overrightarrow{(\forall x)\phi}(\hat{y}), \text{dom}_\phi(\hat{x}) \Rightarrow \overrightarrow{\phi}(\hat{y}, x)$
- (6) $\overrightarrow{(\exists x)\phi}(\hat{y}) \Rightarrow \text{dom}_\phi(\hat{x}), \overrightarrow{\phi}(\hat{y}, x)$
- (7) $\overrightarrow{p(\hat{t})}(\hat{x}) \Rightarrow p(\hat{T})$ for predicate p , where terms \hat{T} correspond to terms \hat{t} with appropriate variable renaming

When a fore translation is encountered at (4), it either completes without exception, and the rules it produces using translations from Fig. 2 are included in the converted theory, or an exception arises and the corresponding rules are discarded, and then the aft version of (4) is used to obtain the converted rules.

Figure 1: **Aft translations**

$$(1) \overleftarrow{\phi}(\hat{x}), \overleftarrow{\psi}(\hat{x}) \Rightarrow \overleftarrow{\phi \wedge \psi}(\hat{x})$$

$$(2) \overleftarrow{\phi}(\hat{x}) \Rightarrow \overleftarrow{\phi \vee \psi}(\hat{x})$$

$$\overleftarrow{\psi}(\hat{x}) \Rightarrow \overleftarrow{\phi \vee \psi}(\hat{x})$$

$$(3) \text{ a) } \overleftarrow{\neg\phi}(\hat{x}) \Rightarrow \overleftarrow{\neg(\phi \wedge \psi)}(\hat{x})$$

$$\overleftarrow{\neg\psi}(\hat{x}) \Rightarrow \overleftarrow{\neg(\phi \wedge \psi)}(\hat{x})$$

$$\text{b) } \overleftarrow{\neg\phi}(\hat{x}), \overleftarrow{\neg\psi}(\hat{x}) \Rightarrow \overleftarrow{\neg(\phi \vee \psi)}(\hat{x})$$

$$\text{c) } \overleftarrow{\phi} \Rightarrow \overleftarrow{\neg\neg\phi}$$

$$\text{d) } \overleftarrow{\phi}(\hat{x}), \overleftarrow{\neg\psi}(\hat{x}) \Rightarrow \overleftarrow{\neg(\phi \rightarrow \psi)}(\hat{x})$$

$$\text{e) } \mathbf{EXCEPTIONAL} : \dots??? \Rightarrow \overleftarrow{\neg(\exists\hat{x})\phi}(\hat{y})$$

$$\text{f) } \overleftarrow{(\exists x)\neg\phi}(\hat{y}) \Rightarrow \overleftarrow{\neg(\forall\hat{x})\phi}(\hat{y})$$

$$\text{g) } \text{not } a(\hat{X}) \Rightarrow \overleftarrow{\neg a}(\hat{x}) \text{ when } a \text{ is atomic}$$

$$(4) \overleftarrow{\neg\phi}(\hat{x}) \Rightarrow \overleftarrow{\phi \rightarrow \psi}(\hat{x}).$$

$$\overleftarrow{\psi}(\hat{x}) \Rightarrow \overleftarrow{\phi \rightarrow \psi}(\hat{x})$$

$$(5) \mathbf{EXCEPTIONAL} : \dots??? \Rightarrow \overleftarrow{(\forall\hat{x})\phi}(\hat{y})$$

$$(6) \overleftarrow{\phi}(\hat{y}) \Rightarrow \overleftarrow{(\exists\hat{x})\phi}(\hat{y})$$

$$(7) p(\hat{T}), \text{dom}(\hat{y}) \Rightarrow \overleftarrow{p(\hat{t})}(\hat{x}) \text{ for predicate } p, \text{ where terms } \hat{T} \text{ correspond to terms } \hat{t} \text{ with appropriate variable renaming, and } \hat{y} \text{ consists of variables in } \hat{x} \text{ which do not appear in } \hat{T}.$$

Figure 2: **Fore translations**

In essence, the exceptional cases involve the obligation to provide a fore translation for a universally quantified wff. There is no simple coherent translation for this case. It is clear that both translation methods are sound. We illustrate the required complication in rule (7) using another example.

Example 4. Suppose that the axiom is $\top \rightarrow (\forall x)(\forall y)(p(x) \rightarrow q(x,y))$ and notice that the antecedent of the formal implication does not contain the universal variable y .

$$\begin{aligned}
 & true \Rightarrow \overrightarrow{(\forall x)(\forall y)(p(x) \rightarrow q(x,y))}. \\
 & \overrightarrow{(\forall x)(\forall y)(p(x) \rightarrow q(x,y))}, dom(X), dom(Y) \Rightarrow \overrightarrow{p(x) \rightarrow q(x,y)}(X, Y). \\
 & \overrightarrow{p(x) \rightarrow q(x,y)}(X, Y), \overleftarrow{p(x)}(X, Y) \Rightarrow \overrightarrow{q(x,y)}(X, Y). \\
 & * \overrightarrow{p(x)}(X, Y), dom(Y) \Rightarrow \overleftarrow{p(x)}(X, Y). \\
 & \overrightarrow{q(x,y)}(X, Y) \Rightarrow q(x,y).
 \end{aligned}$$

Notice at (*) how $dom(Y)$ is needed in the antecedent to universally quantify Y ; otherwise the occurrence in the consequent would be interpreted as an existential variable.

3 Completeness issues

In order to illustrate the issues involved with completeness, consider the FOL theory of Fig. 3.

$$\begin{aligned}
 & \top \rightarrow \neg r \\
 & \top \rightarrow (p \vee q \rightarrow r) \\
 & \neg q \rightarrow \perp
 \end{aligned}$$

Figure 3: Small theory

An aft-only translation to geolog is shown in Fig. 4, and a proof tree is displayed in Fig. 5. A for-and-aft translation to geolog is shown in Fig. 6, and the corresponding proof tree is displayed in Fig. 7.

Any geolog tree (or tableaux) that can be generated using an aft-only translation of a FOL theory using schema (4) in Fig. 1 could also be constructed using the fore version of (4), so long as the fore version does not meet an exceptional translation (and in which case the aft version is used anyway). This is illustrated in our example: notice how the left branch in Fig. 5 could be constructed using using contrapositive inferences from above node 5 in Fig. 7 using all available fore rules.

Reference [1] argues that a translation scheme which is equivalent to our aft-only method of translation is complete. Thus, if the FOL theory is valid then the aft-translated coherent theory will yield a proof. Based upon the observations above, the fore-and-aft version will also yield a proof, but often requiring many fewer steps. This plausibility argument needs to be replaced by a more formal proof.

It is important to notice that the translation of $\overleftarrow{\phi}$ which arises in (4) of Fig. 1 does *not* affect the translations of *other* any other rules of the theory. It is only the internal logical structure of ϕ that matters.

$$\begin{aligned}
r, not_r &\Rightarrow false. \\
p, not_p &\Rightarrow false. \\
q, not_q &\Rightarrow false. \\
true &\Rightarrow \overrightarrow{\neg\neg q}. \\
\overrightarrow{\neg\neg q} &\Rightarrow q. \\
true &\Rightarrow \overrightarrow{\neg r}. \\
\overrightarrow{\neg r} &\Rightarrow not_r. \\
true &\Rightarrow \overrightarrow{p \vee q \rightarrow r}. \\
\overrightarrow{p \vee q \rightarrow r} &\Rightarrow \overrightarrow{\neg(p \vee q)} | r. \\
\overrightarrow{\neg(p \vee q)} &\Rightarrow not_p, not_q.
\end{aligned}$$

Figure 4: Aft-only translation to geolog

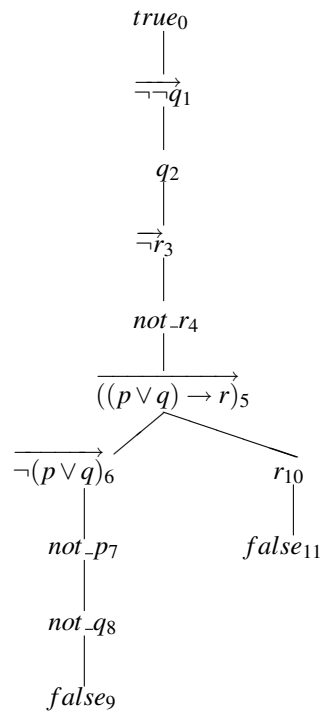


Figure 5: Aft-only proof tree

$$\begin{aligned}
r, not_r &\Rightarrow false. \\
p, not_p &\Rightarrow false. \\
q, not_q &\Rightarrow false. \\
true &\Rightarrow \overleftarrow{\neg\neg}q. \\
\overleftarrow{\neg\neg}q &\Rightarrow q. \\
true &\Rightarrow \overrightarrow{\neg}r. \\
\overrightarrow{\neg}r &\Rightarrow not_r. \\
true &\Rightarrow \overrightarrow{p \vee q \rightarrow} r. \\
\overrightarrow{p \vee q \rightarrow} r, \overleftarrow{p \vee q} &\Rightarrow r. \\
p &\Rightarrow \overleftarrow{p \vee q} \\
q &\Rightarrow \overleftarrow{p \vee q}
\end{aligned}$$

Figure 6: Fore-and-Aft translation to geolog

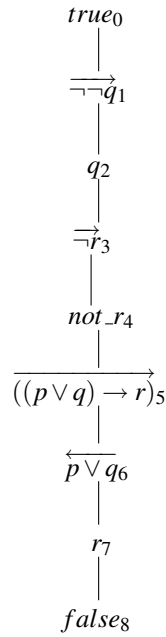


Figure 7: Fore-and-aft

4 Generalizations

4.1 forward translation of universals

It is clear that a coherent FOL formula never raises an exception during fore-and-aft translation of its rules.

It is possible to specify fore translations of universals, provided that we force larger models. Rather than raising an exception in parts (3e) and (5) of Fig.2, consider the following modified fore translation schemas.

- (3d) $\overleftarrow{\neg\phi}(\hat{s}, \hat{y}) \Rightarrow \overleftarrow{\neg(\exists\hat{x})\phi}(\hat{y})$, where arguments \hat{s} are new constants
- (5) $\overleftarrow{\phi}(\hat{s}, \hat{y}) \Rightarrow \overleftarrow{(\forall\hat{x})\phi}(\hat{y})$, where arguments \hat{s} are new constants

Figure 8: **Modified Fore** translations

As an example, consider the FOL theory

$$\top \rightarrow (\forall x)p(x)$$

$$\top \rightarrow ((\forall x)p(x) \rightarrow q)$$

$$q \rightarrow \perp$$

Figure 9: *An example with univeral antecedent*

Fig.10 shows a translation using the original fore-aft schemas, and Fig.11 show a translation using the modified schemas.

$$p(x), \text{not } p(x) \Rightarrow \text{false}$$

$$q, \text{not } q \Rightarrow \text{false}$$

$$\text{true} \Rightarrow \text{not}_q$$

$$\text{dom}(x) \Rightarrow p(x)$$

$$\text{true} \Rightarrow \text{dom}(x), \text{not } p(x) \mid q$$

Figure 10: **AFT** translations

In Fig. 11 *witness* represents a witness for the the fore translation of the universal in the antecedent of the formula.

$$\begin{aligned}
p(x), \text{not}_p(x) &\Rightarrow \text{false} \\
q, \text{not}_q &\Rightarrow \text{false} \\
\text{true} &\Rightarrow \text{dom}(\text{witness}) \\
\text{true} &\Rightarrow \text{not}_q \\
\text{dom}(x) &\Rightarrow p(x) \\
\overleftarrow{(\forall x)p(x)} &\Rightarrow q \\
p(\text{witness}) &\Rightarrow \overleftarrow{(\forall x)p(x)}
\end{aligned}$$

Figure 11: **FORE** translation of universal

The translation in Fig.11 requires a nonempty universe (includes *witness*) and supports the conclusion $(\exists x)p(x)$, whereas the theory in Fig.10 does NOT support the conclusion $(\exists x)p(x)$, and has empty models.

Generally speaking, using the fore rules for all implications may generate a coherent theory which requires the possession of nonempty models, whereas using for-and-aft translation does not force this requirement.

4.2 forwarding disjuncts

The fore-and-aft kind of analysis can be used for disjunctions as well. For example, one could attempt the translation $\overrightarrow{\phi \vee \psi}, \overleftarrow{\neg\phi} \Rightarrow \overrightarrow{\psi}$ rather than $\overrightarrow{\phi \vee \psi}, \Rightarrow \overrightarrow{\phi} | \overrightarrow{\psi}$. However, one then needs to systematically use *other* rules in the theory to be sure to allow inferences eventually having consequent $\overleftarrow{\neg\phi}$. This involves a kind of *contrapositives analysis*.

For example, consider the FOL theory

- (1) $\top \rightarrow p$
- (2) $\top \rightarrow (p \rightarrow q \vee r)$
- (3) $\top \rightarrow (q \rightarrow s)$
- (4) $\top \rightarrow (r \rightarrow s)$
- (5) $s \rightarrow \perp$

One possible extended translation using fore-and-aft analysis for the disjunction in (2) is the following (where several translation steps are combined for brevity).

- (1) $\text{true} \Rightarrow p.$
- (2) $p, \text{not}_q, \text{not}_r \Rightarrow \text{false}$
- (3) $\text{not}_s \Rightarrow \text{not}_q$
- (4) $\text{not}_s \Rightarrow \text{not}_r$

(5) $true \Rightarrow not_s$

Notice that when q and r are both contraposed in (2) this requires q and r to be contraposed in (3) and (4). (Other possible translations could contrapose just one of either q or r .)

Andrew Polonsky has developed a heuristic translation scheme that takes this kind of approach, and this is mentioned in [3]. His approach involves analysis which avoids disjunctions in a general way, and not just the avoidance of disjunction when translating formal implications.

References

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