Box Search
for Coherent Logic

technote

San Clemente, Fall 2009

Abstract. Box search is a partially complete strategy for computing proofs and finite counter models in coherent logic theories using depth-first search. The box terminology refers to a hypothetical box, specified as a depth times a width, where the depth limit restricts the length or depth of each branch the width limit restricts the allowed number of branches. Proofs in boxes generally requires QEDF methods, but extraction of finite counter models only requires a simpler depth-first approach.

1 Examples

The coherent theory in Fig.1

```
true => p(1), p(2).
p(X), p(Y) => q(X,Y,Z), p(Z). // existential rule
q(2,A,B) => goal.
```

Fig. 1. Theory 1

has a proof which fits inside a box of depth ≥ 10 and width ≥ 1, graphically depicted in Fig.2. This small example suffices to establish the necessity of earliest-first satisfaction when using box search for a proof of a coherent theory. Reference [4] explains QEDF, and uses this same example to illustrate the need for earliest-first rule satisfaction for existential rules. It is also true that 10x1 is the smallest box that can contain the QEDF proof.

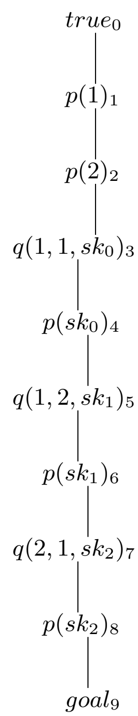

Fig. 2. Proof requires QEDF

The coherent theory in Fig.3 has the counter model $\{q\}$ which is depicted in Fig.4. The branches at leaves 4 and 9 are actually infinite but they were not extended because of the branch depth bound. This example illustrates that counter models can be computed in a depth-first fashion, even if earlier branch are infinite. The smallest box containing this counter model has dimension 2x3. Note that a 2x2 box would not even allow the first rule to fire.

```
true => p(1) | q | p(2).
p(X) => p(f(X)).
```

Fig. 3. Theory 2 has counter model

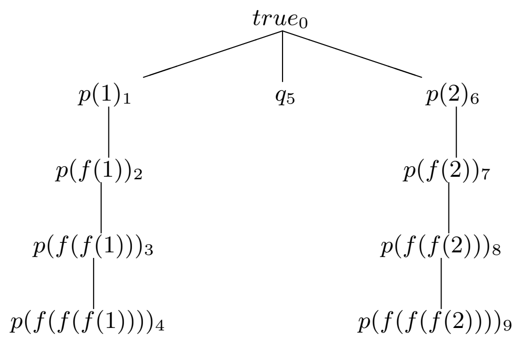

Fig. 4. Counter model for Theory 2 at 5

2 Partial completeness of box search

Lemma 1. If a coherent logic theory has a QEDF proof then some boxed QEDF search finds the proof.

Lemma 2. Suppose that coherent logic theory C has a finite counter model M. Then there is some boxed DF (or QEDF) search which computes M.

Lemma 1 is obvious: One simply selects a box large enough to contain the known proof and QEDF will stay inside that box during search. A proof of Lemma 2 uses a similar observation. Any counter model is an unsuccessfully saturated branch of a geolog tree. If a depth-first search procedure can process that branch it will eventually saturate it, regardless of whether earliest-first satisfaction is used. To guarantee that the search reaches that branch one merely has to set a depth bound deep enough to span the branch and wide enough to include the branch.

One always obtains a bounding box by only specifying a depth bound. Suppose that the theory has a rule with maximum splitting s. Then the largest possible width of an s-tree (every node has s descendants except leaves) at depth d is $N = (s^{d+1} - 1)/(s - 1)$ and the search tree fits inside this d-by-N box.

3 Box metrics

Partial completeness is a somewhat meagre result. Are there heuristic methods to predict *how deep* and *how wide* one needs to search in order to produce either a proof or desired counter models?

References

1. John Fisher and Marc Bezem, Skolem Machines and Geometric Logic. In C.B. Jones, Z. Liu and J. Woodcock, *Proc. ICTAC 2007 The 4th International Colloquium on Theoretical Aspects of Computing*, Macao SAR, China, September 26-28, 2007. Springer LNCS vol. 4711, pp. 201-215.
2. John Fisher and Marc Bezem, Query Completeness of Skolem Machine Computations. In J. Durand-Losé and M. Margenstern, editors, *Proc. Machines, Computations and Universality '07*, Universite d'Orleans - LIFO, Orleans, France September 10-14, 2007. Springer LNCS vol. 4664, pp. 182-192.
3. John Fisher and Marc Bezem, Skolem Machines, *Fundamenta Informaticae*, 91 (1) 2009, pp.79-103.
4. QEDF technote, http://JohnRFisher.NET/colog/qedf.pdf.

[last update 24 November 2009]